TRANSVERSE SHEAR IN LAMINATE ANALYSIS

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ABSTRACT: *This paper describes the method for transverse shear analysis to be applied in the composites design program ESAComp. The correlation of the method to the formulations used in the commercial finite element programs is presented and the results from the selected method are compared to the exact elasticity results. The results show good agreement with the exact solutions for cross-ply and unsymmetric angle-ply laminates. Symmetric angle-ply laminates create bending-twisting coupling that the theory cannot accurately model.*

1 – INTRODUCTION

Composite laminates have weak resistance to transverse shear compared to traditional materials because all reinforcements are normally in the plane of the laminate. Due to the relatively small transverse shear stiffness, the effect of shear deformation becomes significant in structural problems. In addition, transverse shear stresses can cause delamination failure.

In order to account for the shear deformation, the shear stiffness of the laminate has to be determined. The calculation of shear stresses and stiffnesses is much more complicated for composite laminates than for metals because of the layered structure formed from orthotropic plies. In contrast to the three material properties of isotropic metals, orthotropic layers have nine material properties and the material properties may be different in each layer. Since the exact calculation is not possible in a general case, different approximative shear deformation theories have been developed. These theories and their applicability to the composites design program ESAComp[1] were investigated through a literature survey[2].

ESAComp uses Classical Laminate Theory (CLT) in laminate analyses and First order Shear Deformation Theory (FSDT) in plate and beam analyses.[3] The laminate lay-ups and stiffness properties can be exported from ESAComp to the FE-programs. After the analysis is completed, the element resultant stresses can be imported from the FE-programs back to ESAComp for further analyses. The following programs are supported: ABAQUS, ANSYS, I-DEAS, MSC.Nastran and NISA. In order to assure the validity of the results, the compatibility of the selected method to formulations in different FE-programs need to be checked.

1.1 Background

Shear deformation theories typically rely on a displacement field approximation. Theories related to a linear in-plane displacement field across the laminate thickness are called First Order Shear Deformation Theories (FSDT). According to the strain-displacement relation, linear displacement field is analogous to constant transverse shear strains across the laminate thickness. This assumption results in a discontinuous, piecewise constant shear stress field if the shear stresses are calculated from the shear strains. This is unrealistic because the transverse shear stress field should be continuous through the thickness and vanish at the laminate surfaces.

Higher order theories i.e. theories relying on a displacement function higher than first order, have been developed to better describe the internal stress state of the laminate. They can provide a shear stress field that vanishes at the laminate surfaces, but they cannot provide a continuous shear stress field across the laminate thickness.
Both of these conditions can be satisfied by assuming a displacement field separately for each layer. These layerwise theories can provide a powerful tool for transverse shear stress calculations, but they are also computationally challenging and their finite element applications require complicated element formulation.[4]
In practice, displacement fields are significantly curved only when the laminate is very thick. As the thickness of the laminate is decreased, the curved shape gets flatter and the displacement field is thus closer to the FSDT assumption. When the thickness is further decreased, the effect of shear becomes negligible and the displacement field simplifies to the form assumed in the Classical Plate Theory.

2 –TRANSVERSE SHEAR CALCULATION IN ESACOMP

ESAComp laminate analysis is a point analysis, i.e. the analysis is done on a point in the structure with no information about the adjacent points. Basic results given by the analysis are in-plane, bending and shear stiffnesses of a laminate. In the laminate load response analysis the user can specify mechanical, thermal and moisture loads and ESAComp calculates the layer stresses/strains and margins to failure. ESAComp also uses the laminate point analysis as a part of its analysis capabilities for structural elements, such as plates, and for FE postprocessing.
The method for transverse shear chosen for ESAComp needs to provide a calculation method for transverse shear stiffness matrix and for transverse shear stresses. Due to the nature of the point analyses, the shear stiffnesses need to be calculated as a material property independent on the loading conditions and the transverse shear stresses have to be calculated straight from the shear forces.
A method based on the FSDT was chosen to be applied. The FSDT provides calculation methods for transverse shear stiffness matrix and if the transverse shear stresses are calculated from the equilibrium equations and not from the shear strains, the results are quite accurate. Higher order theories cannot be applied because the method needs to be compatible with the FSDT formulations typically used in FE analyses of shell structures. Additionally, a point analysis does not provide enough information for higher order theory applications. The plate analysis of ESAComp would provide this information but in light of consistency the same method was applied in both plate and point analysis.

2.1 Transverse shear stress calculation

The method applied was developed at DLR in Braunschweig. A detailed description of the method can be found in the references[5]. Only the fundamental assumptions and the formulations are presented here.
The shear stresses are obtained by integrating the in-plane stress derivatives with respect to the thickness coordinate (Equation 1.). The theory makes an assumption that the in-plane derivatives of the in-plane forces are zero and that the derivatives $M_{x,x}$ and $M_{y,y}$ represent the shear forces Q_x and Q_y, respectively. All other moment derivatives are assumed to be zero (Equation 2.).

$$\tau_{xz} = -\int \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) dz \qquad [1.]$$

$$\begin{gathered} N_{x,x}=N_{x,y}=N_{y,x}=N_{y,y}=N_{xy,x}=N_{xy,y}=0 \\ M_{x,y}=M_{y,x}=M_{xy,x}=M_{xy,y}=0 \end{gathered} \qquad [2.]$$

Equations for the shear stresses take the form:

$$\begin{Bmatrix} \tau_{xz}{}^{(k)} \\ \tau_{yz}{}^{(k)} \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{32} \\ F_{31} & F_{22} \end{bmatrix}_{(k)} \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} \qquad [3.]$$

The components F_{ij} are second order functions of the thickness coordinate z provided by the 3 x 3 matrix $F(z)$:

$$\begin{aligned} F(z)^{(k)} &= -m(z)[b] - n(z)[d] \\ m(z) &= \int \left[\overline{Q}\right]_{(k)} dz \\ n(z) &= \int \left[\overline{Q}\right]_{(k)} z dz \end{aligned} \qquad [4.]$$

where $[b]$ and $[d]$ are the compliance matrices for coupling and bending, respectively. $\overline{Q}^{(k)}$ is the layer stiffness matrix in the global coordinate system.
This theory uses the three-termed equilibrium equations. Some theories leave out the in-plane shear component. This approach is equivalent to Equation 3 with $F_{32}=F_{31}=0$.

2.2 Shear stiffness calculation

Currently, the shear stiffnesses in ESAComp are calculated using shear correction factors[6]. The shear correction factors are calculated using the principle of strain energy with an assumption that the laminate has symmetric and balanced lay-up. A more precise calculation with no restrictions on the laminate lay-up will be implemented. The calculation of improved shear stiffnesses is based on Reference 7.
The strain energy in matrix terms has two definitions:

$$\{W\} = \frac{1}{2}\int \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^T \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} dz = \frac{1}{2}\int \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^T \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz \qquad [5.]$$

$$\{W\} = \frac{1}{2}\{Q\}^T [K]^{-1} \{Q\} \qquad [6.]$$

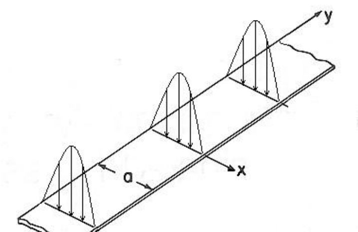
where $\{Q\}$ is the shear force vector and $[K]$ is the shear stiffness matrix.
Substituting the shear stresses presented in the previous chapter to Equation 5 and setting it equal to Equation 6 yields the improved laminate shear stiffnesses K:

$$[K] = \left[\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} F_{11} & F_{32} \\ F_{31} & F_{22} \end{bmatrix}_{(k)}^T \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}_{(k)}^{-1} \begin{bmatrix} F_{11} & F_{32} \\ F_{31} & F_{22} \end{bmatrix}_{(k)} dz \right]^{-1} \qquad [7.]$$

The improved shear stiffnesses account for the actual shear stress field of the laminate and therefore do not need further correction factors.

3 – COMPARISON WITH THE EXACT ELASTICITY SOLUTIONS

The results based on the theory presented in the previous chapter were compared to the exact elasticity solutions by Pagano[8,9]. The exact elasticity solutions are derived for cylindrical bending under a sinusoidal load as shown in Figure 2. The transverse shear stress fields are plotted at the laminate edge (*x=0*).



$q = q_0 \sin(\pi x/a)$

Figure 2 - Cylindrical bending under sinusoidal load

Three simple laminates were examined: cross-ply 0/90/0, unsymmetric angle-ply 15/-15 and symmetric angle-ply 30/-30/-30/30.
The shear force vectors that are given as an input to the calculated field are derived from the plate equations of the Classical Plate Theory (CPT)[10]. The shear forces for the stress calculation become:

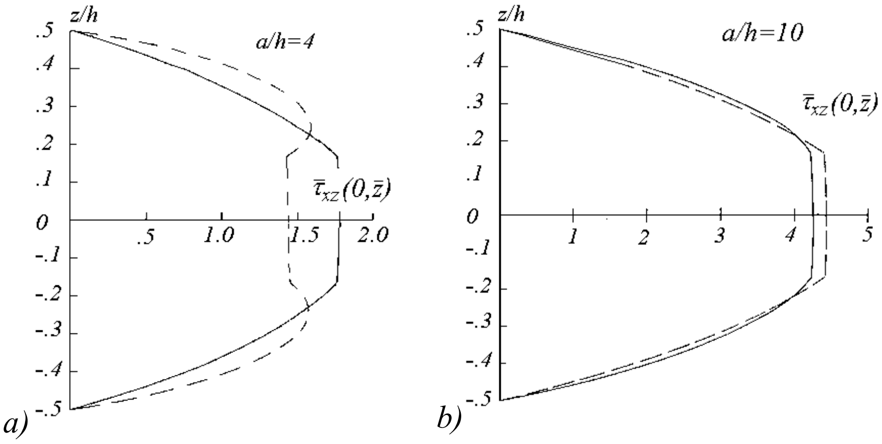
$$\begin{aligned} Q_x &= \frac{dM_x}{dx} + \frac{dM_{xy}}{dy} = \frac{q_0 a}{\pi} \\ Q_y &= \frac{dM_y}{dy} + \frac{dM_{xy}}{dx} = \frac{q_0 a}{\pi} \cdot K_1 \end{aligned} \qquad [8.]$$

were K_1 is defined:

$$K_1 = \frac{B_{16}A_{66}B_{11} - A_{16}B_{16}{}^2 + B_{66}A_{11}B_{16} - B_{66}A_{16}B_{11} - D_{16}A_{11}A_{66} + D_{16}A_{16}{}^2}{-D_{11}A_{11}A_{66} + D_{11}A_{16}{}^2 + A_{66}B_{11}{}^2 - 2B_{11}A_{16}B_{16} + A_{11}B_{16}^2} \qquad [9.]$$

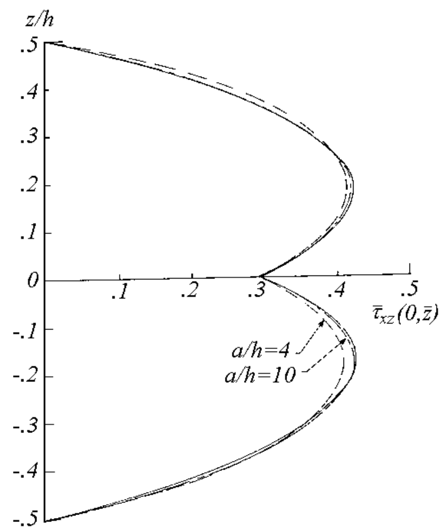
Figure 3a shows the shear stress field of the cross-ply laminate with *a/h*-ratio of four. The exact solution[8] for the shear stress is presented by the dashed line and the calculated result is the solid line. The calculated result shows a shear stress field that is simpler than the exact solution but as the dimensions of the plate are increased, the approximated stress field approaches the exact solution as can be seen in Figure 3b. In this case the shape of the shear stress field depends also on the laminate thickness.
In general, the theories relying on a continuous displacement function across the entire thickness of the laminate cannot estimate the exact shape of the stress field for thick cross-ply laminates. It has also been shown that the higher order theories provide a shear stress field that is different from the exact solution even with high *a/h*-ratios.[11] For this case, the results from the selected theory approaches the exact solution with decreasing thickness. The shear stress field of the thick cross-ply laminate can only be modeled with layerwise theories or the zig-zag theory. The zig-zag theory[12] by DiSciuva says that the displacement field across the thickness is piecewise linear. The laminate curvature is defined separately for each layer and it additionally depends on the ratio of the shear stiffness properties of adjacent layers. For cross-ply laminates with unidirectional plies this parameter is at its maximum and thus generates a shear stress field that cannot be modeled with a linear displacement field.

Figure 4 shows the shear stress field of an unsymmetric angle-ply laminate under the cylindrical bending. The results are additionally divided by the *a/h*-ratio, allowing the results with different *a/h*-ratios to be shown in the same picture. The solid line represents the calculated stress field and the dashed lines provide the exact elasticity results[9]. It can be seen that the calculated results are very close to the exact solution with all *a/h*-ratios.



$$\bar{\tau}_{xz} = \frac{\tau_{xz}(0,z)}{q_0}$$

Figure 3 - Shear stress for [0/90/0] -laminate with *a/h*-ratio *a*) 4 and *b*) 10



$$\bar{\tau}_{xz} = \frac{\tau_{xz}(0,z)}{a/h \cdot q_0}$$

Figure 4 - Shear stress $\tau_{xz}/(a/h)$ for [15/-15] –laminate

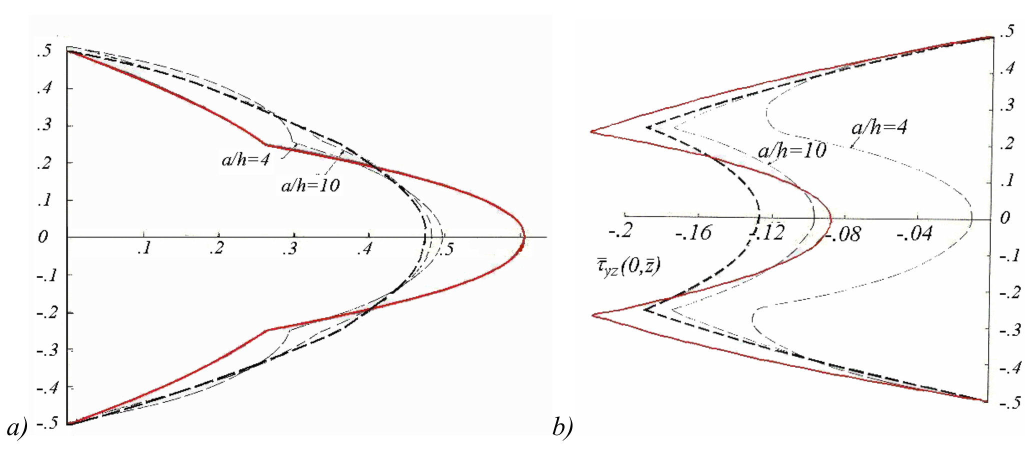


Figure 5 - Shear stress a) $\tau_{xz}/(q_0\ a/h)$ b) $\tau_{yz}/(q_0\ a/h)$ for symmetric [30/-30/-30/30] –laminate

The symmetric angle-ply laminates generate stresses in both transverse planes under one-dimensional cylindrical bending. This is due to the fact that the symmetric angle-ply laminates have nonzero bending stiffness components D_{16} and D_{26} (See Equation 9.). These stiffness terms represent the bending-twisting coupling, which means that the laminate twists as it is bent.
The shear stress fields for the symmetric 30/-30/-30/30 –laminate are shown in Figure 5. The calculated results are presented with solid line and the exact elasticity results[9] with various a/h-ratios are presented with dashed lines. The thick dashed line is the shear stress field that the exact elasticity solution approaches when the a/h-ratio is further increased. It can be seen that the calculated results for τ_{xz} and τ_{yz} deviate from the exact result even with high a/h-ratios. This is because the simplification about the moment derivatives does not hold for the symmetric angle-ply laminates as well as for crossply and unsymmetric angle-ply laminates. In this case the derivatives $M_{y,x}$ and $M_{xy,x}$ have nonzero values. The non-zero moment derivatives for the symmetric angle-ply laminate in the cylindrical bending (for $x=0$) according to the Classical Plate Theory are:

$$M_{x,x} = \frac{q_0 a}{\pi}$$
$$M_{y,x} = 0.3191\ \frac{q_0 a}{\pi} \qquad [10.]$$
$$M_{xy,x} = 0.3978\ \frac{q_0 a}{\pi}$$

If these moment derivatives are used with the DLR method, the results for shear stresses will align with the thick dashed line in Figure 5.

4 – FE – FORMULATIONS

The FE-formulations of the shell elements supported by ESAComp FE-postprocessing were investigated for the transverse shear. These elements include:
- S3R, S4R and S8R for ABAQUS[13]
- SHELL91 and SHELL99 for ANSYS[14] (also SHELL181 in ESAComp 2.1)
- Thin shells for I-DEAS[15]
- DQUAD4 and CTRIA3 for MSC.Nastran[16]

It was discovered that these software, excluding ANSYS, use a symbolic equation similar to Equation 3 to calculate the transverse shear stresses. The formulations vary depending on the software but they all use the simplified form of equilibrium equations which ignore the in-plane shear term:

$$\underline{\text{FE-formulation}} \qquad \underline{\text{Equilibrium equation}}$$
$$\tau_{xz} = \int \frac{\partial \sigma_x}{\partial x} dz \qquad \tau_{xz} = \int \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) dz \qquad [11.]$$

The shear stiffnesses are calculated from the shear stress field approximations by using strain energy principle. The shear stiffness formulations also differ depending on the software. In general they are simplified versions from the DLR version (Equation 7.)

ANSYS, on the other hand, does not print out the shear stiffnesses for laminates. ANSYS uses the incremental form of the 3D equilibrium equations for the shear stress calculations:

$$
\begin{aligned}
\tau_{xz}{}^{k} &= \sum_{i=1}^{k} \Delta\tau_{xz}{}^{i} - \frac{1}{h}\sum_{i=1}^{N} \Delta\tau_{xz}{}^{i} \sum_{i=1}^{k} t^{i} \\
\Delta\tau_{xz}{}^{i} &= -\Delta z\left(\frac{\Delta\sigma_{x}{}^{i}}{\Delta x} + \frac{\Delta\tau_{xy}{}^{i}}{\Delta y}\right)
\end{aligned}
\qquad [12.]
$$

The ANSYS shell elements SHELL91 and SHELL99 have four integration points, where the in-plane stresses are computed. The variation of the in-plane stresses, $\Delta\sigma_{xy}$ and $\Delta\tau_{xy}$, are used in Equation 12. The terms Δx and Δy are the distances between the integration points and Δz is the thickness of the layer. These stresses are called interlaminar shear stresses in ANSYS and they are always given in the layer interfaces. ANSYS also gives stresses called transverse shear stresses that can differ from interlaminar stresses. Transverse shear stresses are calculated from the stress-strain relation and corrected by a second order function to satisfy the free surface boundary conditions. The interlaminar shear stresses are more accurate for most applications.[14]

5 –COMPARISON WITH THE FE-RESULTS

Numerical comparisons were made with the FE-software and DLR formulation. The purpose of the comparisons was not to validate the calculation method but to compare the results of these various formulations. The aforementioned elements were used and a simple loading corresponding to a unit shear force was tested. The ply properties used in calculations were:

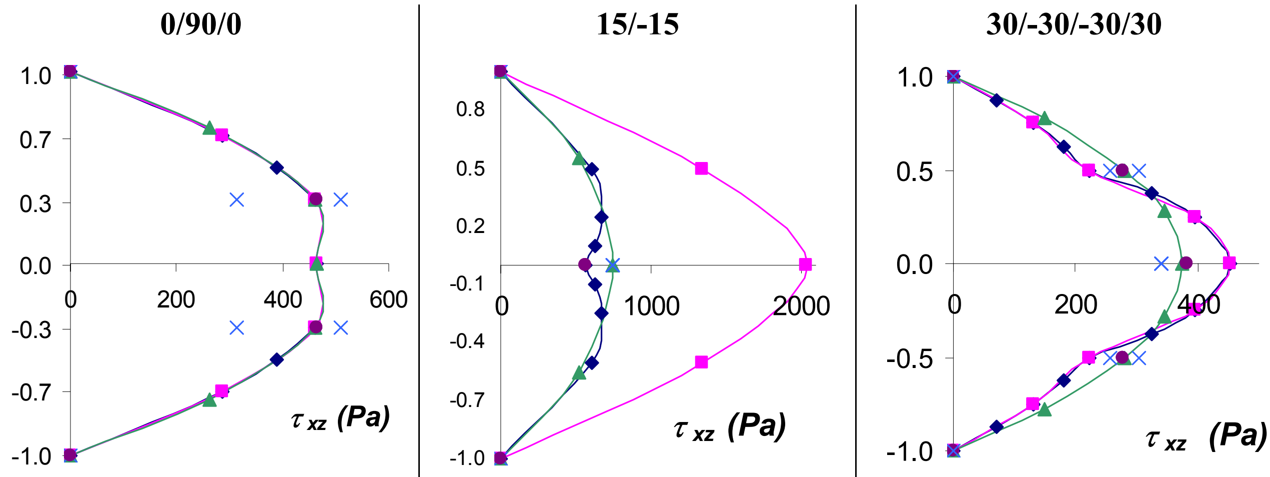
$E_1 = 155GPa \quad E_2 = 8.5GPa \quad G_{12}=G_{13}=5.5GPa \quad G_{23} = 3.4GPa \quad \nu_{12} = 0.3 \quad t = 1mm$

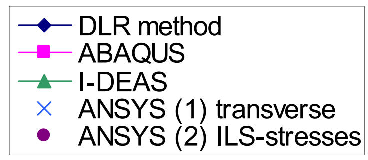
Three laminates were tested:

- three-layer symmetric cross-ply 0/90/0
- four-layer symmetric angle-ply 30/-30/-30/30
- two-layer unsymmetric angle-ply 15/-15

The loading condition corresponding to a unit force ($Q_x = 1N$) was constructed in the Laminate task in I-DEAS. In ABAQUS a model consisting of one element was created. One side was clamped and three sides were free and a line load *1N/m* was applied on the edge opposite to the clamped side. A one-element mesh is acceptable in this case because according to the formulations the shear stresses only depend on the shear forces. The same loading was used in ANSYS as in ABAQUS but a finer mesh was used. A one-element mesh would result in erroneous shear stresses in ANSYS because ANSYS uses displacements when calculating shear stresses. The results were evaluated in the middle of the plate and for the symmetric angle-ply laminate the results were additionally scaled to correspond to a unit force.
The shear stiffnesses and the shear stress fields are presented in Figure 6. The ANSYS result from the stress-strain relations is noted as ANSYS (1) and ANSYS (2) is the result from the incremental form of the equilibrium equations.

Shear stress τ_{xz} corresponding to a unit shear force ($Q_x = 1N$)





Shear stiffness in *MN/m*

0/90/0

$$K_{ABAQUS} = \begin{bmatrix} 10.45 & 0 \\ 0 & 9.12 \end{bmatrix}$$

$$K_{I-DEAS} = \begin{bmatrix} 10.44 & 0 \\ 0 & 9.13 \end{bmatrix}$$

$$K_{DLR} = \begin{bmatrix} 10.45 & 0 \\ 0 & 9.12 \end{bmatrix}$$

15/-15

$$K_{ABAQUS} = \begin{bmatrix} 1.41 & 0 \\ 0 & 5.12 \end{bmatrix}$$

$$K_{I-DEAS} = \begin{bmatrix} 8.80 & 0 \\ 0 & 5.82 \end{bmatrix}$$

$$K_{DLR} = \begin{bmatrix} 8.82 & 0 \\ 0 & 5.78 \end{bmatrix}$$

30/-30/-30/30

$$K_{ABAQUS} = \begin{bmatrix} 14.66 & 1.95 \\ 1.95 & 12.56 \end{bmatrix}$$

$$K_{I-DEAS} = \begin{bmatrix} 16.12 & -1.73 \\ -1.73 & 12.71 \end{bmatrix}$$

$$K_{DLR} = \begin{bmatrix} 13.94 & 2.73 \\ 2.73 & 12.46 \end{bmatrix}$$

Figure 6 – Shear stress fields and shear stiffnesses provided by the FE-software and DLR formulation

In case of cross-ply laminate, the shear stress fields for all programs are aligned and the shear stiffnesses are equal. The stress-strain relation, ANSYS(1), has different results in the interface of two layers depending on which layer is used for the results but the more accurate ANSYS (2) is exactly aligned with the DLR result and with the results from other FE-software.
In case of angle-ply laminates, the results vary depending on the formulation. ABAQUS result for symmetric angle-ply laminate is exactly aligned with the DLR result but the ABAQUS result for the unsymmetric angle-ply laminate differs from all other formulations. This unusually high shear stress field could not be explained by the theoretical background.
The ANSYS (2) equilibrium result agrees with the DLR result for the cross-ply and the unsymmetric angle-ply laminate but for the symmetric angle-ply laminate the ANSYS (2) result is closer to the I-DEAS result. On the other hand, I-DEAS gives parabolic distribution for all cases, which is not true for the unsymmetric angle-ply laminate.

6 – SUMMARY AND CONCLUSIONS

The method to be applied in ESAComp for transverse shear analyses of laminates is based on the three-termed equilibrium equations and the strain energy principle. The results from the selected method were compared to the exact elasticity solutions and the FE-results in case of one-dimensional bending.
The accuracy of the method was good for cross-ply laminates with moderate *a/h*-ratios and for unsymmetric angle-ply laminates with all *a/h*-ratios. For symmetric angle-ply laminates, the shape of the shear stress field did not match with the exact elasticity solution even with high *a/h*-ratios. This is due to the non-zero moment derivatives that are assumed to be zero in the theory. The derivatives are dependent on the coupling terms of the stiffness matrix, which decrease when the number of layers is increased. It can thus be assumed that the accuracy of the method increases as the number of layers is increased.
The FE-formulations are based on a similar approach where the shear stresses are calculated from the shear forces, except for ANSYS. ANSYS uses the incremental form of equilibrium equations and takes advantage of the calculated in-plane stresses in the four integration points of the element. The ANSYS results were closest to the exact solutions for the investigated three laminate lay-ups. ABAQUS gave the same results than the method to be applied in ESAComp except for the unsymmetric angle-ply laminate for which ABAQUS gave surprisingly high shear stresses and low shear stiffnesses that cannot be realistic. I-DEAS formulation is the most simplified but it was closest to the ANSYS result for the symmetric angle-ply laminate.
It can be concluded that the method can be used in FE-postprocessing because the formulations of the FE-programs are similar. In FE-programs the shear stresses are also calculated from the shear forces. When the shear forces are imported to ESAComp for postprocessing, the accuracy of the final result will depend on the accuracy of the imported shear force value. Since the shear stiffnesses have variation among the software, it can be expected that the imported shear forces will differ depending on the software. In ANSYS the results from the incremental form of the equilibrium equations can be more accurate than the results from the selected method because the incremental form also uses the data from adjacent points.

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